Online Appendix

Proposition 2.1 The set $P_n(\mathbb{N})$ forms a group under the operation $+_{\mathrm{P}}$, which adds the debit and credit components separately for each element.

Verification: We verify that the operation $+_{\mathrm{P}}$ is well-defined, implying that the result does not depend on the choice of representatives of the equivalence classes.

Suppose that $\boldsymbol{p}, \boldsymbol{q} \in P_n(\mathbb{N})$ can be written as:

$$\boldsymbol{p} = \left([p_1^d//p_1^c], \dots, [p_n^d//p_n^c]\right)^{\mathrm{t}} = \left([r_1^d//r_1^c], \dots, [r_n^d//r_n^c]\right)^{\mathrm{t}},$$

$$\boldsymbol{q} = \left([q_1^d//q_1^c], \dots, [q_n^d//q_n^c]\right)^{\mathrm{t}} = \left([s_1^d//s_1^c], \dots, [s_n^d//s_n^c]\right)^{\mathrm{t}}.$$

We must check that

$$[p_i^d + q_i^d//p_i^c + q_i^c] = [r_i^d + s_i^d//r_i^c + s_i^c] (i = 1, 2, \dots, n).$$

By the definition of the T-term, $[p_i^d//p_i^c] = [r_i^d//r_i^c]$ implies that $p_i^d + r_i^c = p_i^c + r_i^d$. Similarly, from $[q_i^d//q_i^c] = [s_i^d//s_i^c]$, we have $q_i^d + s_i^c = q_i^c + s_i^d$.

Adding these equalities, we have

$$(p_i^d + q_i^d) + (r_i^c + s_i^c) = (p_i^c + q_i^c) + (r_i^d + s_i^d).$$

By the definition of ∼, this shows

$$[p_i^d + q_i^d//p_i^c + q_i^c] = [r_i^d + s_i^d//r_i^c + s_i^c].$$

Thus, the operation $+_{\mathrm{P}}$ is well-defined.

Proof: To prove that $P_n(\mathbb{N})$ forms a group, the following properties must be verified:

(i) closure, (ii) associativity, (iii) existence of an identity element, and (iv) existence of inverse elements.

Let $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \in P_n(\mathbb{N})$ be arbitrary elements:

$$\boldsymbol{p} = \left([p_1^d//p_1^c], \dots, [p_n^d//p_n^c]\right)^{\mathrm{t}}, \boldsymbol{q} = \left([q_1^d//q_1^c], \dots, [q_n^d//q_n^c]\right)^{\mathrm{t}}, \boldsymbol{r} = \left([r_1^d//r_1^c], \dots, [r_n^d//r_n^c]\right)^{\mathrm{t}}.$$

(i) Closure

By the definition of $+_{\mathrm{P}}$ in $P_n(\mathbb{N})$:

$$\boldsymbol{p} +_{\mathrm{P}} \boldsymbol{q} = \begin{pmatrix} [p_1^d + q_1^d//p_1^c + q_1^c] \\ \vdots \\ [p_n^d + q_n^d//p_n^c + q_n^c] \end{pmatrix}.$$

Since $\boldsymbol{p}, \boldsymbol{q} \in P_n(\mathbb{N})$, we have:

$$\sum_{i=1}^{n} p_i^d = \sum_{i=1}^{n} p_i^c, \sum_{i=1}^{n} q_i^d = \sum_{i=1}^{n} q_i^c.$$

Adding these equalities:

$$\sum_{i=1}^{n}\left(p_i^d + q_i^d\right) = \sum_{i=1}^{n}\left(p_i^c + q_i^c\right)$$

Thus, $\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{q} \in P_n(\mathbb{N})$, proving closure.

(ii) Associativity

$$(\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{q})+_{\mathrm{P}}\boldsymbol{r} = \begin{pmatrix} [p_1^d + q_1^d // p_1^c + q_1^c] \\ \vdots \\ [p_n^d + q_n^d // p_n^c + q_n^c] \end{pmatrix} +_{\mathrm{P}} \begin{pmatrix} [r_1^d // r_1^c] \\ \vdots \\ [r_n^d // r_n^c] \end{pmatrix}$$

$$= \begin{pmatrix} [p_1^d + q_1^d + r_1^d // p_1^c + q_1^c + r_1^c] \\ \vdots \\ [p_n^d + q_n^d + r_n^d // p_n^c + q_n^c + r_n^c] \end{pmatrix}.$$

Similarly,

$$\boldsymbol{p}+_{\mathrm{P}}(\boldsymbol{q}+_{\mathrm{P}}\boldsymbol{r}) = \begin{pmatrix} [p_1^d // p_1^c] \\ \vdots \\ [p_n^d // p_n^c] \end{pmatrix} +_{\mathrm{P}} \begin{pmatrix} [q_1^d + r_1^d // q_1^c + r_1^c] \\ \vdots \\ [q_n^d + r_n^d // q_n^c + r_n^c] \end{pmatrix}$$

$$= \begin{pmatrix} [p_1^d + q_1^d + r_1^d // p_1^c + q_1^c + r_1^c] \\ \vdots \\ [p_n^d + q_n^d + r_n^d // p_n^c + q_n^c + r_n^c] \end{pmatrix}.$$

Thus,

$$(\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{q})+_{\mathrm{P}}\boldsymbol{r} = \boldsymbol{p}+_{\mathrm{P}}(\boldsymbol{q}+_{\mathrm{P}}\boldsymbol{r}).$$

(iii) Identity element

Let $e_i \in \mathbb{N} (i = 1,2,\dots,n)$ and consider the element:

$$\boldsymbol{e} = \begin{pmatrix} [e_1 // e_1] \\ \vdots \\ [e_n // e_n] \end{pmatrix} \in P_n(\mathbb{N}).$$

For any $\boldsymbol{p} = ([d_1//c_1],\dots,[d_n//c_n])^{\mathrm{t}} \in P_n(\mathbb{N})$, consider the i-th component of $\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{e}$:

$$[d_i + e_i // c_i + e_i] = [d_i // c_i].$$

As this is equal to the i-th component of $\boldsymbol{p}$, it follows that:

$$\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{e} = \boldsymbol{p}$$

Similarly, as $\boldsymbol{e}+_{\mathrm{P}}\boldsymbol{p} = \boldsymbol{p}$ also holds, $\boldsymbol{e}$ is the identity element in $P_n(\mathbb{N})$.

Although the identity element $\boldsymbol{e}$ may appear to have multiple representations, any T-term with equal debits and credits is equal to the zero-term. Hence,

$$\boldsymbol{e} = \begin{pmatrix} [e_1 // e_1] \\ \vdots \\ [e_n // e_n] \end{pmatrix} = \begin{pmatrix} [0//0] \\ \vdots \\ [0//0] \end{pmatrix},$$

and the identity element is unique.

(iv) Inverse element

For any $\boldsymbol{p} = ([d_1//c_1],\dots,[d_n//c_n])^{\mathrm{t}} \in P_n(\mathbb{N})$, define its inverse $\boldsymbol{p}^{\mathrm{inv}}$ by swapping the debit and credit components:

$$\boldsymbol{p}^{\mathrm{inv}} = ([c_1//d_1],\dots,[c_n//d_n])^{\mathrm{t}}.$$

Then, we compute:

$$\boldsymbol{p} +_{\mathrm{P}} \boldsymbol{p}^{\mathrm{inv}} = \boldsymbol{p}^{\mathrm{inv}} +_{\mathrm{P}} \boldsymbol{p}$$

$$= \begin{pmatrix} [d_1 + c_1 // d_1 + c_1] \\ \vdots \\ [d_n + c_n // d_n + c_n] \end{pmatrix}$$

$$= \boldsymbol{e}$$

Thus, $\boldsymbol{p}^{\mathrm{inv}}$ is the inverse element of $\boldsymbol{p}$, completing the proof. ■

Proposition 2.2 The set $\mathrm{Bal}_n(\mathbb{Z})$ forms a group under the component-wise addition operation $+_{\mathrm{B}}$.

Proof: To prove that $\mathrm{Bal}_n(\mathbb{Z})$ is a group, we must verify the following properties:

(i) Closure: The sum of any two elements of $\mathrm{Bal}_n(\mathbb{Z})$ remains in $\mathrm{Bal}_n(\mathbb{Z})$.

(ii) Associativity: The addition operation satisfies the associative property.

(iii) Identity element: An element exists in $\mathrm{Bal}_n(\mathbb{Z})$ that acts as an additive identity.

(iv) Inverse element: Each element in $\mathrm{Bal}_n(\mathbb{Z})$ has an additive inverse.

Let $\boldsymbol{u} = (u_1, \dots, u_n)^{\mathrm{t}}, \boldsymbol{v} = (v_1, \dots, v_n)^{\mathrm{t}}, \boldsymbol{w} = (w_1, \dots, w_n)^{\mathrm{t}}$ be arbitrary elements of $\mathrm{Bal}_n(\mathbb{Z})$.

(i) Closure under addition

By the definition of addition in $\mathrm{Bal}_n(\mathbb{Z})$:

$$\boldsymbol{u} +_{\mathrm{B}} \boldsymbol{v} = (u_1 + v_1, \dots, u_n + v_n)^{\mathrm{t}}.$$

Taking the sum of the components:

$$\sum_{i=1}^{n} (u_i + v_i) = \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} v_i$$

$$= 0 + 0$$

$$= 0.$$

Therefore, $\boldsymbol{u} +_{\mathrm{B}} \boldsymbol{v} \in \mathrm{Bal}_n(\mathbb{Z})$, proving closure.

(ii) Associativity

We verify the associativity of $+_{\mathrm{B}}$:

$$(\boldsymbol{u} +_{\mathrm{B}} \boldsymbol{v}) +_{\mathrm{B}} \boldsymbol{w} = (u_1 + v_1, \dots, u_n + v_n)^{\mathrm{t}} + (w_1, \dots, w_n)^{\mathrm{t}}$$

$$= (u_1 + v_1 + w_1, \dots, u_n + v_n + w_n)^{\mathrm{t}}.$$

Similarly,

$$\boldsymbol{u} +_{\mathrm{B}} (\boldsymbol{v} +_{\mathrm{B}} \boldsymbol{w}) = (u_1, \dots, u_n)^{\mathrm{t}} + (v_1 + w_1, \dots, v_n + w_n)^{\mathrm{t}}$$

$$= (u_1 + v_1 + w_1, \dots, u_n + v_n + w_n)^{\mathrm{t}}.$$

As both expressions are equal, associativity holds:

$$(\boldsymbol{u}+_{\mathrm{B}}\boldsymbol{v})+_{\mathrm{B}}\boldsymbol{w} = \boldsymbol{u}+_{\mathrm{B}}(\boldsymbol{v}+_{\mathrm{B}}\boldsymbol{w}).$$

(iii) Identity element

Define $\mathbf{0}_n$ as the zero vector $(0,\dots,0)^{\mathrm{t}} \in \mathrm{Bal}_n(\mathbb{Z})$. For any $\boldsymbol{u} \in \mathrm{Bal}_n(\mathbb{Z})$:

$$\boldsymbol{u}+_{\mathrm{B}}\mathbf{0}_n = \mathbf{0}_n+_{\mathrm{B}}\boldsymbol{u} = (u_1,\dots,u_n)^{\mathrm{t}}.$$

Thus, $\mathbf{0}_n$ is the identity element in $\mathrm{Bal}_n(\mathbb{Z})$.

(iv) Inverse element

For any $\boldsymbol{u} = (u_1,\dots,u_n)^{\mathrm{t}} \in \mathrm{Bal}_n(\mathbb{Z})$, define $\boldsymbol{u}^{\mathrm{inv}}$ as:

$$\boldsymbol{u}^{\mathrm{inv}} = (-u_1,\dots,-u_n)^{\mathrm{t}} \in \mathrm{Bal}_n(\mathbb{Z}).$$

Then,

$$\boldsymbol{u}+_{\mathrm{B}}\boldsymbol{u}^{\mathrm{inv}} = \boldsymbol{u}^{\mathrm{inv}}+_{\mathrm{B}}\boldsymbol{u} = \mathbf{0}_n.$$

Thus, $\boldsymbol{u}^{\mathrm{inv}}$ is the additive inverse of $\boldsymbol{u}$.

As $\mathrm{Bal}_n(\mathbb{Z})$ satisfies closure, associativity, the existence of an identity element, and the existence of inverse elements, it forms a group under the addition operation $+_{\mathrm{B}}$.■

Theorem 3.1 The Pacioli group $P_n(\mathbb{N})$ and the balance module $\mathrm{Bal}_n(\mathbb{Z})$ are isomorphic as groups.

Verification: We verified that f is well-defined, that is, the mapping f sends elements of $P_n(\mathbb{N})$ to the same element of $\mathrm{Bal}_n(\mathbb{Z})$, independent of the choice of representatives.

Suppose an element $\boldsymbol{p} \in P_n(\mathbb{N})$ is expressed using different representatives as follows:

$$\boldsymbol{p} = \begin{pmatrix} [d_1//c_1] \\ \vdots \\ [d_n//c_n] \end{pmatrix} = \begin{pmatrix} [d_1'//c_1'] \\ \vdots \\ [d_n'//c_n'] \end{pmatrix}.$$

Focusing on the i-th component, we have:

$$[d_i//c_i] = [d_i'//c_i'].$$

By the definition of the T-term, the following equality holds:

$$d_i + c_i' = d_i' + c_i.$$

Interpreting this as an equality in $\mathbb{Z}$, we rearrange it to obtain:

$$d_i - c_i = d_i' - c_i'.$$

Thus, it follows that:

$$f\left(\begin{pmatrix} [d_1//c_1] \\ \vdots \\ [d_n//c_n] \end{pmatrix}\right) = \begin{pmatrix} d_1 - c_1 \\ \vdots \\ d_n - c_n \end{pmatrix} = \begin{pmatrix} d_1' - c_1' \\ \vdots \\ d_n' - c_n' \end{pmatrix} = f\left(\begin{pmatrix} [d_1'//c_1'] \\ \vdots \\ [d_n'//c_n'] \end{pmatrix}\right).$$

Therefore, we have confirmed that f is well-defined.

Proof: To prove the isomorphism, we show that the mapping $f: P_n(\mathbb{N}) \to \mathrm{Bal}_n(\mathbb{Z})$ is a group homomorphism and a bijection (both surjective and injective).

Step 1: Proving that f is a group homomorphism

Group homomorphism is a mapping from one group to another that is known to "preserve the group structure." In this paper, it refers to a mapping satisfying the following property: the balance vector (an element of the balance module) obtained by applying f to the sum of two elements of the Pacioli group is equal to the sum of the balance vectors obtained by applying f to each element individually. In other words, to show that f is a group homomorphism, we must show that for any $\boldsymbol{p},\ \boldsymbol{p}' \in P_n(\mathbb{N})$, the following holds:

$$f(\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{p}') = f(\boldsymbol{p})+_{\mathrm{B}}f(\boldsymbol{p}').$$

Now we suppose that

$$\boldsymbol{p} = \begin{pmatrix}[d_1//c_1]\\ \vdots \\ [d_n//c_n]\end{pmatrix}, \boldsymbol{p}' = \begin{pmatrix}[d'_1//c'_1]\\ \vdots \\ [d'_n//c'_n]\end{pmatrix}.$$

We compute the image of their sum under f:

$$f(\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{p}') = f\left(\begin{pmatrix}[d_1//c_1]\\ \vdots \\ [d_n//c_n]\end{pmatrix}+_{\mathrm{P}}\begin{pmatrix}[d'_1//c'_1]\\ \vdots \\ [d'_n//c'_n]\end{pmatrix}\right).$$

By the definition of $+_{\mathrm{P}}$ in $P_n(\mathbb{N})$:

$$= f\left(\begin{pmatrix}[d_1 + d'_1//c_1 + c'_1]\\ \vdots \\ [d_n + d'_n//c_n + c'_n]\end{pmatrix}\right).$$

Applying the definition of f:

$$= \begin{pmatrix}d_1 + d'_1 - c_1 - c'_1\\ \vdots \\ d_n + d'_n - c_n - c'_n\end{pmatrix}.$$

Rewriting:

$$= \begin{pmatrix}d_1 - c_1\\ \vdots \\ d_n - c_n\end{pmatrix}+_{\mathrm{B}}\begin{pmatrix}d'_1 - c'_1\\ \vdots \\ d'_n - c'_n\end{pmatrix}.$$

Since $f(\boldsymbol{p}) = (d_1 - c_1, \dots, d_n - c_n)^{\mathrm{t}}$ and $f(\boldsymbol{p}') = (d'_1 - c'_1, \dots, d'_n - c'_n)^{\mathrm{t}}$, we conclude:

$$f(\boldsymbol{p}+_{\mathrm{P}}\boldsymbol{p}') = f(\boldsymbol{p})+_{\mathrm{B}}f(\boldsymbol{p}').$$

Thus, f is a group homomorphism.

Step 2: Proving that f is surjective

Let $\boldsymbol{v} = (v_1, \dots, v_n)^t \in \mathrm{Bal}_n(\mathbb{Z})$ be an arbitrary element. Define a list of T-terms $\boldsymbol{p}^{\boldsymbol{v}}$ such that its i-th component p_i^v is given by:

$$p_i^v = \begin{cases} [v_i//0] & \text{if } v_i \geq 0, \\ [0//-v_i] & \text{if } v_i < 0. \end{cases}$$

Next, we compute the total debit sum S^{debit} and total credit sum S^{credit}:

$$S^{\text{debit}} = \sum_{i \text{ s.t.} v_i \geq 0} v_i \,, S^{\text{credit}} = -\sum_{i \text{ s.t.} v_i < 0} v_i.$$

As $\boldsymbol{v}$ is an element of $\mathrm{Bal}_n(\mathbb{Z})$, it satisfies:

$$\sum_{i=1}^{n} v_i = \sum_{i \text{ s.t.} v_i \geq 0} v_i + \sum_{i \text{ s.t.} v_i < 0} v_i = 0.$$

Rearranging:

$$\sum_{i \text{ s.t.} v_i \geq 0} v_i = -\sum_{i \text{ s.t.} v_i < 0} v_i.$$

Thus, the total debit and credit sums are equal, implying that $\boldsymbol{p}^{\boldsymbol{v}} \in P_n(\mathbb{N})$.

Finally, applying f to $\boldsymbol{p}^{\boldsymbol{v}}$:

$$f(\boldsymbol{p}^{\boldsymbol{v}}) = \boldsymbol{v}.$$

Because for every element $\boldsymbol{v} \in \mathrm{Bal}_n(\mathbb{Z})$, a corresponding $\boldsymbol{p}^{\boldsymbol{v}} \in P_n(\mathbb{N})$ exists such that $f(\boldsymbol{p}^{\boldsymbol{v}}) = \boldsymbol{v}$, the mapping f is surjective.

Step 3: Proving that f is injective

To prove injectivity, we must show that for any two elements $\boldsymbol{p}$ and $\boldsymbol{p}'$:

$$\boldsymbol{p} = ([d_1//c_1], \dots, [d_n//c_n])^{\mathrm{t}}, \boldsymbol{p}' = ([d_1'//c_1'], \dots, [d_n'//c_n'])^{\mathrm{t}},$$

$f(\boldsymbol{p}) = f(\boldsymbol{p}')$ implies $\boldsymbol{p} = \boldsymbol{p}'$.

If $f(\boldsymbol{p}) = f(\boldsymbol{p}')$, then for each i:

$$d_i - c_i = d_i' - c_i'.$$

Rewriting:

$$d_i + c_i' = d_i' + c_i.$$

By the definition of the T-term equivalence relation, this implies:

$$[d_i//c_i] = [d_i'//c_i'].$$

Thus, $\boldsymbol{p} = \boldsymbol{p}'$, proving that f is injective.

As $f: P_n(\mathbb{N}) \to \mathrm{Bal}_n(\mathbb{Z})$ is a bijective group homomorphism, it is a group isomorphism. Therefore, the Pacioli group, $P_n(\mathbb{N})$ and the balance module, $\mathrm{Bal}_n(\mathbb{Z})$ are isomorphic as groups. ■