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**Declaration of interest**

The author declares no conflicts of interest relevant to the content of this article.

**Declaration of generative AI in scientific writing**

During the preparation of this study, the author used ChatGPT 4o and Gemini 2.0 Flash to proofread the English text. After using these tools/services, the author reviewed and edited the content as necessary and takes full responsibility for the content of this article.

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# Isomorphism Between Two Algebraic Structures in Mathematical Models of Double-Entry Accounting Systems: The Pacioli Group and the Balance Module

**Keito Ozawa**

Waseda University

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**Abstract**

This study investigates the mathematical properties of double-entry accounting systems by demonstrating the isomorphism between the Pacioli group and the balance module. Although both structures serve as formal models of double-entry bookkeeping and share algebraic properties, their mathematical relationships remain unexplored. Using group homomorphism, this study demonstrates that the Pacioli group and the balance module are isomorphic, implying that they can be transformed into each other while preserving their structural properties. This finding allows for the cross-application of previous research findings between these models and provides a unified mathematical perspective on double-entry bookkeeping. The results contribute to the theoretical advancement of accounting research and potentially influence the design of accounting systems.

**Keywords**: Double-entry bookkeeping, Algebraic structures, Pacioli group, Balance module, Isomorphism

Corresponding Author: Keito Ozawa, Waseda University, Graduate School of Commerce, 1-6-1 Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, JAPAN

Email: keito.ozawa@akane.waseda.jp

## 1. Introduction

Mathematical research on double-entry accounting systems is limited, and their structure remains largely unexplored. Since Mattessich's (1957, 1964) works, the axiomatic approach to accounting research has garnered attention for its role in clarifying the fundamental assumptions of accounting and their consequences by eliminating the influence of conventions and interpretations (Balzer & Mattessich, 2000). However, most axiomatic studies have focused solely on the duality of double-entry bookkeeping, and as Ellerman (2014) highlighted, they have not attempted to mathematically capture the underlying structure of bookkeeping within double-entry accounting systems.

This study examines the mathematical properties of double-entry accounting systems by focusing on two algebraic structures used in their formalization: the Pacioli group, introduced by Ellerman (1982), and the balance module, studied by Rambaud et al. (2010). Both belong to a well-known class of algebraic structures known as groups. Intuitively, in an additive group, elements can be added together, and for every element, another element exists that cancels it out, effectively allowing for subtraction (see Lang (2002) for a formal definition). The Pacioli group comprises ordered pairs of natural numbers and models the T-form of the general ledger. The balance module comprises integer vectors whose components sum to zero and models a double-entry accounting system with trial balances and journal entries. Although both algebraic structures form the foundation of the mathematical models of double-entry accounting systems and are categorized as groups, their mathematical relationships have not been explicitly established.

This study investigates whether the Pacioli group and the balance module can be compared using group homomorphisms, a key concept in algebra[1]. Group homomorphism is a mapping that preserves a group's mathematical structure while transferring it to another group. Because both the Pacioli group and the balance module share the property that the set of journal entries forms a group, we investigate whether a group homomorphism exists between them and, if so, what its characteristics are.

This study demonstrates the isomorphism between the Pacioli group and the balance module. In other words, although the Pacioli group and the balance module appear distinct, they can be transformed into one another while preserving their respective computational structures. This finding enables the results of prior research on one algebraic system to be transferred to the oth-

[1] A homomorphism simplifies computationally complex problems, enabling efficient processing. For instance, rather than directly measuring the distance from Tokyo Skytree to Sensō-ji, a map-based approach can be used. On a map, movement is represented as vectors, and distances are determined by measuring vector lengths and applying a scale factor. When combining the vector from Tokyo Skytree to Sensō-ji with the vector from Sensō-ji to Ueno, the result is the vector from Tokyo Skytree to Ueno. This preserves the vector addition rule. Thus, a homomorphism transforms complex computations into a simpler structure while maintaining fundamental operational rules.

er. As detailed in Section 4, the isomorphism theorem has the potential to accelerate the practical adoption of directed graphs of journal entries (Arya et al., 2000; Rambaud et al., 2010) and fraud detection methods based on graph analysis (Guo et al., 2022; Liang, 2023). The theorem also provides developers the freedom to select system architectures while ensuring data completeness and reliability (Barra et al., 2010).

The remainder of this paper is organized as follows: Section 2 defines the two algebraic structures and reviews previous research. Section 3 establishes the isomorphism theorem between the Pacioli group and the balance module and interprets its implications. Section 4 discusses the practical implications of the isomorphism theorem. Finally, Section 5 concludes the paper and discusses future research directions.

## 2. Two Algebraic Structures

In this section, we describe the definitions and properties of the Pacioli group and the balance module. Throughout this paper, we consider a set of accounts $A = \{a_1, a_2, \cdots, a_n\}$. As the foundation for the two algebraic systems, we adopt the set of natural numbers $\mathbb{N} = \{0,1,2,\dots\}$ or the set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, following most examples presented in Rambaud et al. (2010). The standard addition for $\mathbb{N}$ or $\mathbb{Z}$ is denoted by $+$.

### 2.1 Pacioli Group

Ellerman (1982) constructed an algebraic structure known as the group of differences (Bourbaki, 1989, p. 20) using ordered pairs of natural numbers. This structure serves as the foundation of his mathematical model of double-entry accounting systems. These ordered pairs mathematically represent the debit and credit totals of a T-form in double-entry bookkeeping. For a given account, with debit total $d$ and credit total $c$, the T-account is modeled as an ordered pair $(d//c) \in \mathbb{N} \times \mathbb{N}$ (Figure 1). The use of natural numbers (i.e., non-negative numbers) reflects the conventional avoidance of negative numbers in double-entry bookkeeping. In this paper, the first component of an ordered pair is referred to as the debit, while the second component is referred to as the credit.

Account $a_i$

| | | | | | |
|---|---|---|---|---|---|
| Transaction 1 | $x$ | Transaction 2 | $y$ | | |
| ⋮ | ⋮ | ⋮ | ⋮ | | |
| Debit Total | $d$ | Credit Total | $c$ | ↔ | $(d//c)$ |

T-form in the general ledger — T-form represented by an ordered pair

**Figure 1: Correspondence between T-forms in the general ledger and their representation as an ordered pair.**

The Pacioli group was obtained by identifying T-forms that produced the same account balance. For any $m \in \mathbb{N}$, a T-form represented by $(100//80)$ and one represented by $(100+m//80+m)$ both yield the same debit balance of 20. This reflects the fundamental property of double-entry bookkeeping: entries of equal amounts on both the debit and credit sides do not affect the balance. The Pacioli group is defined as the set of ordered pairs satisfying this equivalence rule, with all T-forms balancing the total debits and credits.

**Definition 2.1** (T-term and Pacioli Group) An equivalence relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$$(x//y) \sim (w//z) \Leftrightarrow x + z = y + w.$$

The equivalence class of $(d//c)$, denoted as $[d//c]$, is called a T-term. The set of all T-terms is denoted by $C(\mathbb{N})$. Let $C_n(\mathbb{N})$ denote the set of vectors whose $i$-th components are $[d_i//c_i]$. The set of vectors in $C_n(\mathbb{N})$ that satisfy the condition $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} c_i$ is denoted by $P_n(\mathbb{N})$ and is referred to as the Pacioli group. That is:

$$P_n(\mathbb{N}) = \left\{([d_1//c_1], [d_2//c_2], \ldots, [d_n//c_n])^{\mathrm{t}} \middle| \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} c_i \right\}.$$

The following proposition describes the algebraic structure of $P_n(\mathbb{N})$, which serves as the basis for referring to it as the Pacioli "group."

**Proposition 2.1** The set $P_n(\mathbb{N})$ forms a group under the operation $+_{\mathrm{P}}$, which adds the debit and credit components separately for each element.

**Proof**: For the case $n = 1$, this has been proven by Ellerman (1985), where the identity element is $[0//0]$, and the inverse of $[d//c] \in P_1(\mathbb{N})$ is $[c//d]$. For the general case of $P_n(\mathbb{N})$, see the online appendix.

### 2.2 Balance Module

The balance module is an algebraic structure comprising integer vectors whose component sums are zero. These vectors model the trial balances and journal entries (Rambaud et al., 2010). The zero-component sum property mathematically represents the equilibrium principle between debits and credits in double-entry bookkeeping. For instance, suppose the trial balance at a given moment is described by three accounts: asset, liability, and equity, with balances represented by $X, Y, Z \in \mathbb{N}$. The balance sheet equation, $X = Y + Z$ holds true for these variables. Rewriting this equation as $X - Y - Z = 0$ highlights that the components of the vector $(X, -Y, -Z)^{\mathrm{t}}$ sum to zero. This idea can be generalized to a system with $n$ accounts, defining the balance vectors and their set, known as the balance module. Both trial balances and journal entries can be represented as balance vectors (Rambaud et al., 2010, pp. 35–37, 51–52).

**Definition 2.2** (Balance Vector and Balance Module) For each account $a_i (i = 1, 2, \ldots, n)$, let the account balance $v_i \in \mathbb{Z}$. A vector $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^{\mathrm{t}}$ satisfying $\sum_{i=1}^{n} v_i = 0$ is called a balance vector. The set of all balance vectors is denoted as $\mathrm{Bal}_n(\mathbb{Z})$, which is referred to as the balance module.

The following proposition is a fundamental statement about the algebraic structure of the balance module, i.e., the sets of trial balances and journal entries.

**Proposition 2.2** The set $\mathrm{Bal}_n(\mathbb{Z})$ forms a group under the component-wise addition operation $+_{\mathrm{B}}$.
**Proof**: Rambaud et al. (2010) defined $\mathrm{Bal}_n(\mathbb{Z})$ as the kernel of a module homomorphism rather than directly proving that $\mathrm{Bal}_n(\mathbb{Z})$ is a group. However, the fact that $\mathrm{Bal}_n(\mathbb{Z})$ is a group directly follows from its definition and the addition operation $+_{\mathrm{B}}$. See the online appendix.

**2.3 Discussion on Prior Research**

Although Ellerman (2014) and Rambaud et al. (2010) cited each other's work, the mathematical relationship between the Pacioli group and the balance module remains unexplored.

Ellerman (2014, pp. 487–488) distinguishes between two systems of double-entry bookkeeping: the "Single-Sided accounts with Signed numbers" (SSS) system and the "Double-Sided accounts with Unsigned numbers" (DSU) system. The former does not separate accounts into debits and credits but instead represents the double-entry accounting system using signed numbers in $\mathbb{Z}$. By contrast, the latter explicitly distinguishes between debits and credits by employing unsigned numbers in $\mathbb{N}$. The balance module corresponds to the SSS system, whereas the Pacioli group corresponds to the DSU system. Ellerman (2014, p. 492) states that the SSS and DSU systems are equivalent as transaction recording systems. However, he did not explicitly discuss the mathematical relationship between the Pacioli group and the balance module, nor did he investigate the existence of homomorphisms between them.

Although Rambaud et al. (2010, pp. 54–55) introduced the Pacioli group, they did not discuss its relationship with the balance module. Ellerman (2014, p. 500) criticized the construction of the Pacioli group for using ordered pairs of integers instead of natural numbers, overlooking the convention of double-entry bookkeeping to avoid negative numbers. Furthermore, Rambaud et al. (2010, p. 55) claimed that the construction of the Pacioli group results in the loss of transaction information; however, this assertion lacks an algebraic discussion.

We discuss this in detail in Section 4 after proving Theorem 3.1 (the isomorphism theorem). The isomorphism $f$ introduced in Section 3 maps a T-term $[d//c]$ to the net balance $d - c$ for each account, but the order of transactions and the detailed history of account changes are omitted. Therefore, both the Pacioli group and the balance module abstract away the same historical information. Moreover, by applying the inverse map $f^{-1}$ to each column of the balance matrix introduced by Rambaud et al. (2010), we can construct a Pacioli group-based *transaction sequence matrix* in which the rows represent accounts, and the columns represent a list of T-terms (Figure 2).

$$\begin{pmatrix} 500 & 400 & 300 & 700 \\ 100 & 100 & -50 & 50 \\ -350 & -300 & -100 & -400 \\ -250 & -200 & -150 & -350 \end{pmatrix} \xrightarrow[\text{to each column}]{\text{applying the group homomorphism}} \begin{pmatrix} [500//0] & [400//0] & [300//0] & [700//0] \\ [100//0] & [100//0] & [0//50] & [50//0] \\ [0//350] & [0//300] & [0//100] & [0//400] \\ [0//250] & [0//200] & [0//150] & [0//350] \end{pmatrix}$$

**Figure 2: Conversion from a balance matrix to a transaction sequence matrix based on the Pacioli group (numerical example from Rambaud et al., 2010, p. 56).**

## 3. Isomorphism between the Pacioli Group and the Balance Module

The Pacioli group and the balance module are isomorphic as groups. First, we present the mathematical theorem and its proof. Next, we provide an interpretation of this theorem.

**Theorem 3.1** The Pacioli group $P_n(\mathbb{N})$ and the balance module $\mathrm{Bal}_n(\mathbb{Z})$ are isomorphic as groups.
**Proof**: We define a mapping $f$ as follows:

$$f: P_n(\mathbb{N}) \ni \begin{pmatrix} [d_1//c_1] \\ \vdots \\ [d_n//c_n] \end{pmatrix} \mapsto \begin{pmatrix} d_1 - c_1 \\ \vdots \\ d_n - c_n \end{pmatrix} \in \mathbb{Z}^n.$$

Because the elements of the domain $P_n(\mathbb{N})$ are lists of equivalence classes, it is necessary to verify that $f$ is well-defined. It can be easily shown that the mapping yields the same element regardless of the choice of the representative. According to the definition of $P_n(\mathbb{N})$, $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} c_i$. This equation, which is valid for $\mathbb{N}$, is also valid for $\mathbb{Z}$. Whereas not every element in $\mathbb{N}$ has an additive inverse, every element in $\mathbb{Z}$ does. Adding the inverse element $-\sum_{i=1}^{n} c_i$ to both sides results in:

$$\sum_{i=1}^{n} (d_i - c_i) = 0.$$

Thus, $(d_1 - c_1, \dots, d_n - c_n)^{\mathrm{t}}$ is a balance vector and $f$ maps $P_n(\mathbb{N})$ to $\mathrm{Bal}_n(\mathbb{Z})$. To complete the proof, it remains to be shown that $f$ is both a group homomorphism and a bijection. See the online appendix.

This theorem demonstrates that the group structure of the double-entry accounting system is universal, whether viewed from the perspective of a trial balance or the general ledger. The function $f$ serves as a bridge between these two perspectives.

The mapping $f$ plays the role of "netting" the total debit and credit amounts for each account to compute its balance, thereby modeling the process of deriving the trial balance from the general ledger. Because $f$ is a group homomorphism, it preserves operations: in the Pacioli group, adding a new transaction (an element of the Pacioli group) to the existing general ledger corresponds to adding the corresponding balance vector (an element of the balance module) to the trial balance.

Moreover, the fact that $f$ is bijective implies that there is a one-to-one correspondence between the Pacioli group and the balance module. In other words, they are equivalent as algebraic structures, which implies that they are structurally identical.

## 4. Practical Implications

The group isomorphism between the Pacioli group and the balance module has two practical consequences. First, it can help accelerate the adoption of graph-based fraud detection methods in accounting and auditing practices. Second, it gives system developers greater freedom in selecting the architecture for accounting information systems while still ensuring data completeness and reliability.

### 4.1 Fraud Detection through Graph Analysis

Arya et al. (2000) proposed visualizing double-entry records as directed graphs, laying the groundwork for recent studies on graph-based fraud detection (Guo et al. 2022; Liang 2023). Rambaud et al. (2010, pp. 69–72, Chapter 4) also described building graphs directly from balance vectors. According to the isomorphism theorem, we can start from a list of T-terms (elements of the Pacioli group) and construct an equivalent directed graph. Because each T-term explicitly encodes debits and credits in the familiar format used by practitioners, it is easier to integrate existing graph-analysis tools into real-world accounting workflows.

### 4.2 Choice of Architecture in Accounting Information Systems

Ellerman's (2014) claim in Section 2.3 that SSS and DSU architectures are equivalent is now formally justified by the isomorphism theorem. Because the debit-credit balancing and zero-sum vector formulations are mathematically interchangeable, system designers may select either an SSS implementation corresponding to the balance module or a DSU implementation corresponding to the Pacioli group, depending on factors such as the ease of implementation. The isomorphism theorem mathematically guarantees that either choice yields an accounting information system with built-in internal controls, thus ensuring data completeness and reliability (Barra et al., 2010).

## 5. Conclusion

This study demonstrated that the Pacioli group and the balance module are isomorphic. This theorem allows research findings to be shared and cross-applied between the two structures. The significance of this study lies in its role in enhancing the understanding of the mathematical structure underlying double-entry accounting systems and in providing a foundation for integrating prior studies. This contribution is expected to advance the theoretical development of accounting while offering practical benefits for the design and operation of accounting systems.

However, several research challenges remain to be addressed. Clarifying the mathematical relationships between other models of double-entry accounting systems beyond the Pacioli group and the balance module is essential, thereby establishing a unified mathematical framework for analyzing double-entry accounting systems. Further, future research should explore the practical benefits that a mathematical approach to accounting systems can offer to accounting practices. Although it has been shown mathematically that either the SSS or DSU system may be adopted, the choice between them when developing accounting information systems should also be informed by considerations from the perspective of information engineering.

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